This question paper contains 4 printed pages.

Your Roll No. ....

I

Sl. No. of Paper	: 91
Unique Paper Code	: 32351302
Name of the Paper	: **Group Theory - I**
Name of the Course	: **B.Sc. (Hons.) Mathematics**
Semester	: **III**
Duration	: **3 hours**
Maximum Marks	: **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

*Attempt any **two** parts from each question.*
***All** questions are compulsory.*

1. (a) Define a group. Give an example of:

   (i) an abelian group consisting of eight elements,

   (ii) a non-abelian group consising of six elements,

   (iii) an infinite abelian group, and

   (iv) an infinite non-abelian group.

   (b) Show that the set {5, 15, 25, 35} is a group under multiplication modulo 40. What is the identity element of this group? Find the inverse of each element.

   (c) Prove that the intersection of an arbitrary family of subgroups of a group G is again a subgroup of G. What can you say about the union of two subgroups? Justify your answer. 2×6=12

P. T. O.

2. (a) (i) Prove that in (Z, +), the group of integers under addition, every non-zero element is of infinite order.

(ii) Let G be a group and $a \in G$. If $|a| = n$ and k is positive divisor of n, then prove that $|a^{n/k}| = k$.

(b) Prove that the order of a cyclic group is equal to the order of its generator.

(c) Define a cyclic group. If $G = (a)$ is a finite cyclic group of order n, then prove that the order of any subgroup of G is a divisor of n, and for each positive divisor k of n, G has exactly one subgroup of order k namely, $(a^{n/k})$.

2×6.5=13

3. (a) Prove that if the identity permutation $\varepsilon = \beta_1 \cdots \beta_r$ where the β's are 2-cycles then r is even.

(b) Show that for $n \geq 3$, $Z(S_n) = \{I\}$.

(c) Prove that:

(i) a group of prime order has no proper, non-trivial subgroup. State its converse. Is it true?

(ii) a group of prime order is cyclic and any non-identity element can be taken as its generator.

2×6=12

4. (a) Let G be a finite group of permutations of a set S. Then prove that for any i from S:

$$|G| = |orb_G(i)|\,|stab_G(i)|.$$

(b) (i) Prove that the center Z(G) of a group G is a subgroup of G and is normal in G.

(ii) If H is a subgroup of G such that H is contained in the center Z(G), then prove that H is a normal subgroup of G. Is the converse true? Justify your answer.

(c) Let N be a normal subgroup of a group G and let H be a subgroup of G. If N is a subgroup of H, prove that H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G. 2×6.5=13

5. (a) Let **C** be the complex numbers and:

$$\mathbf{M} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in R \right\}.$$

Prove that **C** and **M** are isomorphic under addition and $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and $\mathbf{M}^* = \mathbf{M} \setminus \{0\}$ are isomorphic under multiplication.

(b) Prove that an infinite cyclic group is isomorphic to (Z, +). Hence show that every subgroup of an infinite cyclic group is isomorphic to the group itself.

(c) Let G be a group of permutations. For each σ in G, define

$$\operatorname{sgn}(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ is an even permutation,} \\ -1, & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Prove that sgn is a homomorphism from G to $\{1, -1\}$. What is the kernel? 2×6 =12

P. T. O.

6\. (a) Let ϕ be a homomorphism from a group G to a group $\bar{G}$. Let g be an element of G. Then:

(i) $\phi(g^n) = \phi(g)^n$ for all $n \in \mathbf{Z}$.

(ii) ϕ is one-one if and only if $\ker(\phi) = \{e\}$, where e is the identity of G.

(b) State and prove the First Isomorphism Theorem.

(c) (i) Suppose ϕ is a homomorphism from $U(30)$ to $U(30)$ and $\text{Ker}(\phi) = \{1, 11\}$.

If $\phi(7) = 7$, find all elements of $U(30)$ that map to 7.

(ii) Let G be a group. Prove that the mapping $\phi(g) = g^{-1}$, for all $g \in$ G, is an isomorphism from G onto G if and only if G is Abelian. 2×6.5=13